A Mechanical Device
for the
Solution of Equations
involving the
Laplacian Operator

by
GLENN MURPHY
and
J. V. ATANASOFF

IOWA ENGINEERING EXPERIMENT STATION
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THE IOWA ENGINEERING EXPERIMENT STATION
IOWA STATE COLLEGE
AMES, IOWA

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For the Solution of Equations
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ACKNOWLEDGMENTS

The idea of constructing an instrument to assist in the solution of the Laplacian originated with Dr. J. V. Atanasoff, who was at that time Associate Professor of Physics at Iowa State College. The instrument was designed under the auspices of Project 210 of the Iowa Engineering Experiment Station. After preliminary tests established the practicability of the apparatus, space was allocated by the Physics Department for the equipment, and it was used by Mr. L. A. Hannum in his research as a part of the requirement for the Master's degree with a major in Mathematical Physics. Mr. Hannum's thesis, "Approximation to Solutions of Poisson's Equation by Means of Geometrical Models", 1940, is on file at the Iowa State College library.

The equipment was constructed by the College Instrument Shop, under the direction of Mr. I. A. Coleman.

ABSTRACT

This bulletin describes the construction and operation of a device, called the Laplaciometer, which is used in the construction of a surface forming the solution of certain types of second order differential equations in two independent variables. The application of the instrument and method to the solution of two torsion problems indicates that results (stress) within 2 per cent of theoretical values may be obtained. Application to other types of problems is indicated.

The principal advantage of this method of solution over the experimental solution using the membrane analogy is that this method results in the formation of a permanent surface, whereas the soap film usually employed as the membrane is, at best, temporary.

A Mechanical Device For the Solution of Equations Involving the Laplacian Operator

INTRODUCTION

The Laplacian operator in two independent variables, $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, appears in the mathematical expression of several important engineering problems. The solution of a number of problems in the theory of elasticity, in fluid dynamics and in electrical theory involves the solution of an equation (the Poisson equation) in which the Laplacian, operating on a variable, is equal to a constant or to a variable.

Analytical procedures for the solution of these equations generally lead to approximate results because of the complex boundary conditions frequently encountered in engineering problems. The fact that the form of the equation is the same for several problems has made possible the formulation of useful analogies. One example is the membrane analogy (1), by means of which the stresses in a torsional member of irregular (but constant) cross section may be evaluated from observations of the shape of the surface formed by increasing (or decreasing) the pressure on the side of an elastic membrane (usually a soap film) stretched across an opening having the same outline as the cross section of the bar in torsion.

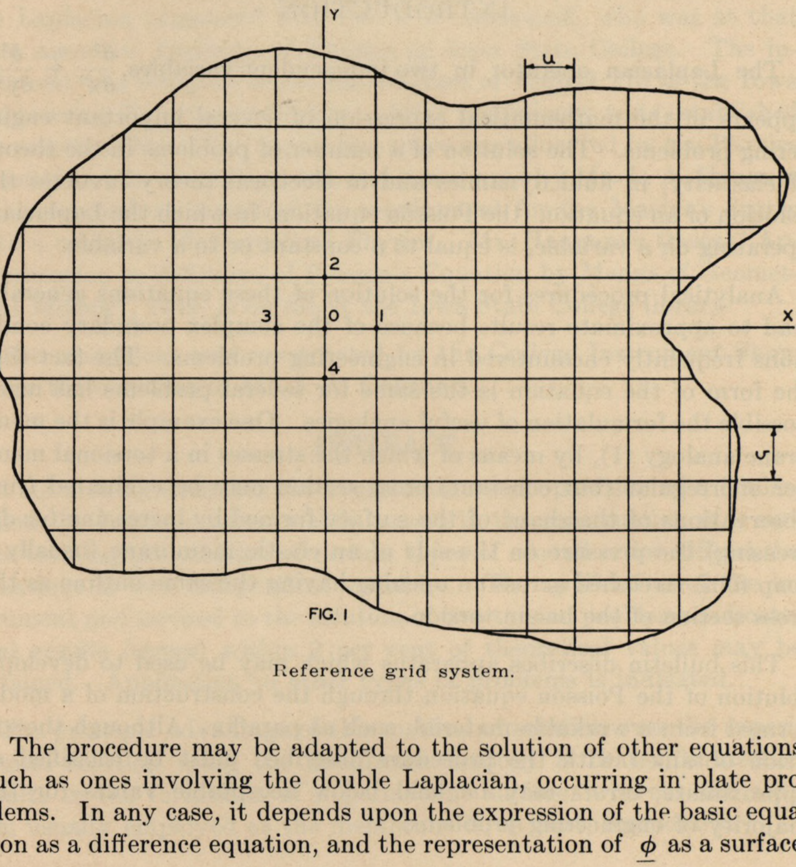
This bulletin describes apparatus which may be used to develop a solution of the Poisson equation through the construction of a model formed from a workable material, such as paraffin. Although the solution obtained with the procedure described must be classified as approximate, errors may be held below acceptable values for the majority of engineering problems.

METHOD

The solution of a problem by the procedure described in this bulletin involves the formation of a surface, using a plastic material such as paraffin, with the aid of an instrument, which will be called a Laplaciometer. The procedure described is particularly adaptable to the solution of an equation of the form

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = F \qquad (1)$$

in which $\underline{\phi}$ is the unknown dependent variable
$\underline{x}$ and $\underline{y}$ are coordinate designations
$\underline{F}$ is a known variable or a known constant, including zero.



Reference grid system.

The procedure may be adapted to the solution of other equations, such as ones involving the double Laplacian, occurring in plate problems. In any case, it depends upon the expression of the basic equation as a difference equation, and the representation of $\underline{\phi}$ as a surface.

THE SURFACE

The difference equation, which approximates the Laplacian on the left-hand side of Eq. (1), may be derived by considering a surface $z = f(x,y)$. An arbitrary rectangular grid is established in the $\underline{xy}$ plane as indicated in Fig. 1. The $\underline{x}$ interval of the grid is designated u, and the y interval, v. Four points (1,2,3,4) surrounding the arbitrary origin are located as indicated. The elevations of the surface at the five points, starting with the origin, are z_0, z_1, z_2, z_3, and z_4.

At any point on the surface, $\frac{\partial^2 z}{\partial x^2}$ is the rate of change of the slope in the x direction or

$$\frac{\partial^2 z}{\partial x^2} = \lim_{u \to 0} \left\{ \frac{\lim\limits_{u \to 0} \dfrac{z_1 - z_0}{u} - \lim\limits_{u \to 0} \dfrac{z_0 - z_3}{u}}{u} \right\} \tag{2}$$

If u does not approach the limit zero, but remains a small finite quantity, an approximate value of $\frac{\partial^2 z}{\partial x^2}$ is

$$\frac{\partial^2 z}{\partial x^2} = \frac{\dfrac{z_1 - z_0}{u} - \dfrac{z_0 - z_3}{u}}{u} \tag{3}$$

$$= \frac{z_1 + z_3 - 2z_0}{u^2} \tag{4}$$

Similarly, an approximate value of $\frac{\partial^2 z}{\partial y^2}$ at any point may be determined as

$$\frac{\partial^2 z}{\partial y^2} = \frac{z_2 + z_4 - 2z_0}{v^2} \tag{5}$$

The Laplacian at any point may therefore be written as (approximately)

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{z_1 + z_3 - 2z_0}{u^2} + \frac{z_2 + z_0 - 2z_0}{v^2} \tag{6}$$

If a square grid is used, u = v and

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{z_1 + z_2 + z_3 + z_4 - 4z_0}{u^2} \tag{7}$$

The error involved in the approximation is discussed in the appendix and for most engineering problems may be reduced to a negligible value by proper selection of the grid, keeping it small in comparison with the area to which the Laplacian applies.

The Laplacian of the surface, z, is equal to some quantity Q, so

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = Q \tag{8}$$

If the surface is laid out so that its projection on the xy plane is geometrically similar to the area over which the original equation, Eq. 1, applies

$$x = nx_1 \tag{9a}$$

$$y = ny_1 \tag{9b}$$

in which n is a constant scale factor.

In addition, if

$$Q = n_1 F \tag{10}$$

it follows that Eq. (8) becomes

$$\frac{1}{n^2}\frac{\partial^2 z}{\partial x_1^2} + \frac{1}{n^2}\frac{\partial^2 z}{\partial y_1^2} = n_1 F \tag{11}$$

or

$$\frac{\partial^2 z}{\partial x_1{}^2} + \frac{\partial^2 z}{\partial y_1{}^2} = n^2 n_1 F \tag{12}$$

Hence, by comparison with Eq. (1),

$$z = n^2 n_1 \phi \tag{13}$$

That is, the surface which is formed represents the unknown function to a different scale, and the desired characteristics of the function (slope, volume, etc.) may be determined from appropriate observations on the surface. The surface becomes, in effect, a model of the original function, ϕ.

THE LAPLACIOMETER

The Laplaciometer is designed to assist in shaping the z surface so that it satisfies the Laplacian and conforms to Eq. (13). A "design" equation may be developed by eliminating the Laplacian between Eqs. (7) and (8).

$$\frac{z_1 + z_2 + z_3 + z_4 - 4z_0}{u^2} = Q \tag{14}$$

Equation (14) combined with Eq. (10) gives

$$z_1 + z_2 + z_3 + z_4 - 4z_0 = u^2 n_1 F \tag{15}$$

which applies at every point on the surface.

If the quantity F is a constant, as is the case in several important problems,

$$u^2 n_1 F = K \tag{16}$$

and Eq. (15) reduces to

$$z_1 + z_2 + z_3 + z_4 - 4z_0 = K \tag{17}$$

The Laplaciometer used in this investigation was designed to satisfy Eq. (17), although it may readily be used for situations in which F is a known function of x and y.

The instrument consists of a frame, Fig. 2, supporting five vertical rods and permitting them to move only in the vertical direction. The rods are spaced to correspond to the five grid points used in the analysis. Rods 1 and 2 are connected to a cross link A and rods 3 and 4 to a cross link B. Links A and B are connected at their mid-points to supplementary rods which actuate link C, all links and rods being slotted to permit free movement. The mid-point of link C may be clamped to rod O, with the dial indicating the relative elevation of O when clamped. The lower end of each rod terminates in a foot which may rest on the surface being shaped.

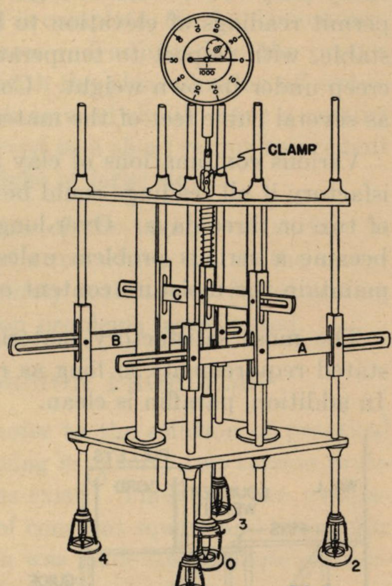


FIG. 2

Sketch of Laplaciometer.

That the elevations of the feet on the five rods satisfy Eq. (17) for any combination of elevations may be shown by assuming the four outside feet to be at zero elevation and considering the effect of a change in elevation of any one of them. With the outside feet at zero elevation, the elevation of foot O is, from Eq. (17) $-\frac{1}{4}$ K. If foot 1 is displaced a distance h, with feet 2, 3, and 4 remaining stationary, the mid-point of link A will move a distance $\frac{1}{2}$ h and the mid-point of link C a distance $\frac{1}{4}$ h causing rod O and foot O to move a distance $\frac{1}{4}$ h. If these values are substituted in Eq. (17) there results

$$(z_1 + h) + z_2 + z_3 + z_4 - 4(z_0 + \tfrac{1}{4}h) = K \qquad (18)$$

which reduces to Eq. (17)

$$z_1 + z_2 + z_3 + z_4 - 4z_0 = K$$

The same result will be obtained regardless of which foot or what combination of feet are moved. Hence, the Laplaciometer satisfies Eq. (17) for all positions, so long as its axis remains vertical.

If the quantity F in Eq. (1) is a variable, the Laplaciometer may still be used. However, the relative height of the center rod must be adjusted from point to point to satisfy Eq. (16) over the entire surface.

The dial on the Laplaciometer indicates the relative elevation of the center rod with respect to the frame. If the four outside rods are at zero elevation, the dial reading $\underline{h}$ corresponds to $\underline{z_0}$ in Eq. (14). Hence,

$$Q = \frac{-4h}{u^2} \tag{19}$$

MATERIALS AND ACCESSORY EQUIPMENT

Since the method involves the formation of a surface which is to represent the unknown variable, consideration must be given to the material of which the surface is developed. The ideal material is easily shaped, but firm enough to support the Laplaciometer and to permit readings of elevation to be taken. It must be dimensionally stable, with respect to temperature and humidity changes, and not creep under its own weight. Cost is also an important consideration as several cubic feet of the material may be required.

Various combinations of clay and sand were used, and proved satisfactory if all readings could be obtained within a total elapsed time of two or three days. Over longer periods, shrinkage of the material became a serious problem unless special precautions were taken to maintain the moisture content of the clay-sand mixture.

The most satisfactory material tried was paraffin. It meets the stated requirements so long as room temperatures are not excessive. In addition, paraffin is clean.

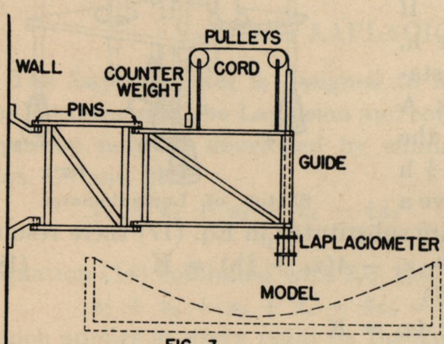


FIG. 3

In Fig. 3 at left is a wall bracket support which permits lateral and vertical movement of the Laplaciometer.

In Fig. 4 a depth gage is in position on the model.

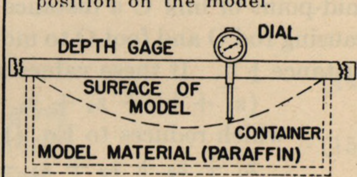


FIG. 4

The container for the surface is essentially a box with an open top. The outline of the box conforms with the outline of the region involved in the problem under consideration. For example, if the method is used to evaluate torsional stresses in a tap, the outline of the container must conform with the outline of the tap, and the top of the container must be a plane in order to satisfy the boundary conditions involved in the torsion problem.

A movable support for the Laplaciometer is very convenient. A bracket of the type indicated in Fig. 3 meets the general requirements of a support which will permit the Laplaciometer to be moved easily, which will support its weight, and which will maintain its axis in the vertical position.

A useful accessory, not illustrated, is a small plane or cutting tool which may be attached to the center rod in place of the foot. This tool is helpful in planing the surface to final shape. After the preliminary shaping (usually done by hand) is complete, the Laplaciometer, with the plane attached and the center rod adjusted to the proper setting, may be run over the surface with the plane shaving off excess material on each pass. Ideally the plane used should have a cutting edge approximately one inch long curved to a short radius to prevent gouging on the side, especially in regions of large slope.

A depth gage for reading elevations of the completed surface is useful. One type of gage consisting of a rod with a dial at the center is indicated in Fig. 4. This was entirely satisfactory for obtaining readings in connection with the torsion problems investigated.

APPLICATIONS TO TORSION PROBLEMS

The adaptability of the Laplaciometer to the solution of practical engineering problems was tested by using it to solve two torsion problems for which mathematical solutions exist. These are the evaluation of stresses in torsional members of constant square and triangular cross sections. In addition, a solution was made for an angle section.

The general differential equation for a bar of constant cross section subjected to a torsional couple at each end is

$$\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial y_1^2} = \frac{-2\,G\,\theta}{L} \qquad (20)$$

in which $\underline{\phi}$ is the "stress function"

$\underline{x_1}$, $\underline{y_1}$ are coordinates in the cross section with respect to an origin at the center of twist

$\underline{G}$ is the modulus of rigidity of the material

$\underline{\theta}$ is the total angle of twist

L is the length of the bar

The components of torsional shearing stress on the cross section may be determined from the stress function as

$$S_{xz} = \frac{\partial \phi}{\partial y_1} \tag{21a}$$

$$S_{yz} = \frac{-\partial \phi}{\partial x_1} \tag{21b}$$

and the torque required to develop the angle of twist Θ in a length L is

$$T = 2\int_0^A \phi \, dx_1 \, dy_1 \tag{22}$$

If the stress function is considered to be plotted along a third coordinate axis against x and y, any component of shearing stress is the slope of the stress function surface in the direction perpendicular to the direction of the stress component. That is,

$$S_{tz} = \frac{\partial \phi}{\partial n} \tag{23}$$

This characteristic makes it possible to establish the boundary conditions of the stress function. At any point on the boundary of the cross section the component of the shearing stress normal to the boundary must be zero because such a component would induce a shearing stress in the longitudinal direction on the free surface, and such a stress cannot exist. From Eq. (23) it follows that the slope of the stress function along the boundary must be zero, and all points on the boundary must lie on a plane. It is convenient to establish the boundary at zero elevation. Then it may be shown from Eq. (22) that the torque is equal to twice the volume under the stress function.

If the shaft has a constant cross section and is homogeneous, the right hand side of Eq. (20) is constant, and a z surface, proportional to ϕ, may be established with the aid of the Laplaciometer. The slope of the z surface is proportional to torsional stress, and the volume is proportional to torque.

The size of the surface to be constructed is governed in part by the size of the Laplaciometer. The instrument used in this investigation was designed to fit a square grid 3 in. on a side. That is, u = 3 in. The base upon which the models were constructed was made approximately 30 in. square as it was thought that a model approximately ten times the size of the grid would be sufficiently large to reduce the effect of the approximation involved in the Laplacian to a conservative value.

The other arbitrary factor to be established is the value of the Laplacian, or the magnitude of the quantity $\underline{Q}$ in Eq. (8). This controls the relative height (or depth) of the surface and is governed by the amount of material available for the model. A value of 0.086 in. was used for the square section reported in this bulletin, and the instrument was arbitrarily adjusted so that the surface was concave. This gave a maximum depth of surface (distance of surface from the plane of the edges) of approximately $2\frac{1}{2}$ inches, which was satisfactory.

A first approximation to the surface may be obtained by casting the paraffin in the container, the edges of which are in a plane, bringing the material in tangent to the edges, and roughing out the center· The analogy between the surface and the shape of a soap film (or other elastic membrane) stretched over an opening of similar outline and distended by pressure on one side is helpful. After the rough shaping the plane is attached to the center rod and wheels to the outer rods of the Laplaciometer and it is run over the surface starting with the corners, where the most elevations are established, and working toward the center. Final shaping with a hand tool is expedient, using the Laplaciometer with feet attached to check the configuration of the surface.

After the surface had been formed for the square shape being investigated, elevations were taken on a 2-in.* grid network using the

TABLE I

Elevations of Surface Corresponding to Square Section

Values of $\underline{z}$ in Inches

y	x	0	2	4	6	8	10	12	14	15⅛
0		2.52	2.48	2.37	2.18	1.91	1.52	1.04	0.41	0
2		2.49	2.47	2.34	2.14	1.88	1.50	1.03	0.41	0
4		2.35	2.32	2.22	2.04	1.82	1.42	0.98	0.37	0
6		2.16	2.13	2.04	1.89	1.65	1.32	0.91	0.34	0
8		1.92	1.90	1.79	1.65	1.34	1.17	0.81	0.32	0
10		1.52	1.50	1.44	1.33	1.17	0.96	0.67	0.25	0
12		1.04	1.03	0.99	0.93	0.83	0.57	0.48	0.20	0
14		0.44	0.44	0.41	0.39	0.36	0.30	0.24	0.12	0
15⅛		0	0	0	0	0	0	0	0	0

*Selected arbitrarily.

depth gage. Average values are given in Table I for one quarter of the symmetrical section. The origin is at the center of the square.

A contour map of the surface is shown in Fig. 5. The contour lines, which are also called "lines of shearing stress" indicate the directions of the maximum shearing stress as well as the elevation of the z surface.

From Table I the maximum slope is seen to occur on the line x = 0 between y = 15 and y = 15 1/8. The magnitude of the maximum slope is approximately

$$\frac{\partial z}{\partial y} = \frac{0.44}{1.125} = 0.391 \qquad (23)$$

From Eqs. (21a) and (13)

$$S_{xz} = \frac{1}{n\,n_1}\,\frac{\partial z}{\partial y} \qquad (24)$$

$$= \frac{0.391}{n\,n_1} \qquad (24a)$$

The constant n in Eq. (24a) is the ratio of the size of the model to the size of the actual bar being twisted. For a bar having a square cross section with length of side a,

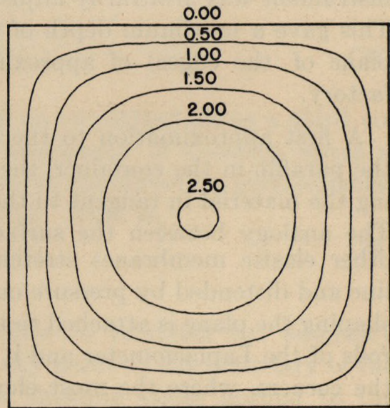


FIG. 5

Contour lines for stress function of square cross-section in torsion. Elevations are shown in Table I.

$$n = \frac{30.25}{a} \qquad (25)$$

The constant n_1 is determined from Eq. (10), in which Q, from Eq. (19) is $\frac{4}{9}$ or 0.0382 in. Hence,

$$n_1 = \frac{0.0382L}{2G\,\theta} \qquad (26)$$

Thus, the stress may be evaluted for any given angle of twist as

$$S_{xz} = 0.677\,\frac{G\theta a}{L} \qquad (27)$$

Timoshenko (4) indicates the theoretical maximum stress to be

$$S_{xz} = 0.6756\,\frac{G\theta a}{L} \qquad (27a)$$

The agreement is reasonably good.

In order to obtain a relationship between stress and torque the volume under the $\underline{z}$ surface must be determined, and the result used in Eq. (22). If $\underline{\phi}$ in Eq. (22) is replaced by its equivalent from Eq. (13)

$$T = \frac{2}{n^2 n_1} \int_0^A z \, dx_1 \, dy_1 \tag{28}$$

$$= \frac{2V}{n^4 n_1} \tag{28a}$$

in which $\underline{V}$ is the volume under the $\underline{z}$ surface.

The volume under the $\underline{z}$ surface was determined for the 30.25-in. square model using Simpson's one-third rule, as 1116 cu. in. Therefore

$$T = \frac{2232}{n^4 n_1} \tag{29}$$

$$= \frac{2232 \, G\theta}{0.0191 \, n^4 L} \tag{29a}$$

$$= 0.1396 \, \frac{G\theta a^4}{L} \tag{29b}$$

The maximum stress may be expressed in terms of torque by eliminating $\frac{G\theta}{L}$ from Eqs. (27) and (29b)

$$S_{xz} = 4.849 \, \frac{T}{a^3} \tag{30}$$

Timoshenko (4) indicates the maximum stress to be

$$S_{xz} = 4.81 \, \frac{T}{a^3} \tag{30a}$$

which compares favorably with the results obtained by the model.

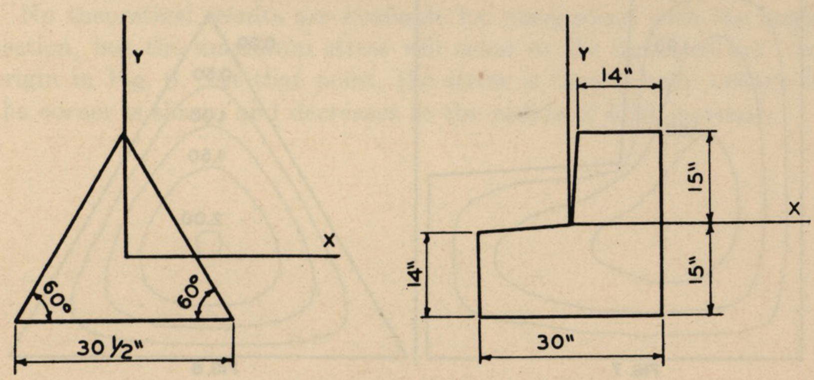


FIG. 6
Dimensions and coordinates for triangle and angle sections.

TABLE II

Elevations of Surface Corresponding to Angle Section

Values of $\bar{z}$ in Inches

y	x	0	1	2	5	7	9	11	13	15
15			0	0	0	0	0	0	0	0
14			—	0.15	0.20	0.25	0.22	0.17	0.09	0
12			0.12	0.44	0.60	0.66	0.63	0.51	0.24	0
10			0.18	0.62	0.77	0.95	0.90	0.70	0.32	0
8			0.25	0.76	1.05	1.14	1.08	0.84	0.39	0
6			0.30	0.87	1.18	1.29	1.21	0.94	0.43	0
4			0.37	0.98	1.31	1.42	1.31	1.01	0.49	0
2			0.48	1.13	1.45	1.54	1.39	1.06	0.51	0
0		0	0.79	1.35	1.60	1.63	1.45	1.10	0.52	0
— 2			1.26	1.57	1.73	1.71	1.50	1.13	0.54	0
— 4				1.76	1.81	1.74	1.51	1.12	0.55	0
— 6					1.82	1.70	1.47	1.10	0.54	0
— 8						1.56	1.34	1.01	0.50	0
—10							1.14	0.84	0.40	0
—12								0.60	0.28	0
—14									0.10	0
—15		0	0	0	0	0	0	0	0	0

Table II shows values of $\bar{z}$ as determined from an angle section, and Table III shows the values of $\bar{z}$ as determined from measurements on a model in the form of an equilateral triangle 30.5 in. on each side. The location of the origin for each is shown in Fig. 6. The reading on the Laplaciometer was 0.177 for the triangle and 0.123 in. for the angle section. Contours of the angle and triangle surfaces are indicated in figures 7 and 8.

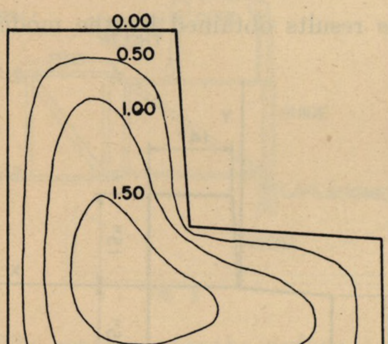


FIG. 7

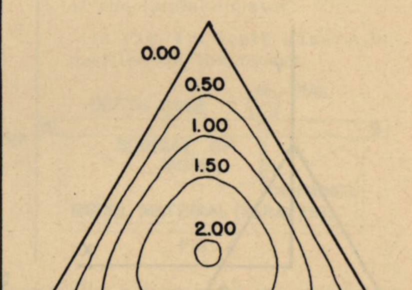


FIG. 8

Contour lines for stress function of angle and triangle cross-sections in torsion. Elevations of angle are shown in Table II; of triangle in Table III.

TABLE III

Elevations of Surface Corresponding to Triangular Section

Values of $\underline{z}$ in Inches

y	x	0	2	4	6	8	10	12	14	15¼
17.6		0								
16.0		0.07								
14.0		0.22								
12.0		0.49	0.32							
10.0		0.83	0.65							
8.0		1.16	1.01	0.56						
6.0		1.47	1.34	0.96	0.28					
4.0		1.75	1.64	1.30	0.76					
2.0		1.95	1.85	1.57	1.09	0.41				
0.0		2.02	1.94	1.71	1.32	0.78				
—2.0		1.94	1.88	1.69	1.39	0.98	0.43			
—4.0		1.66	1.62	1.49	1.28	0.99	0.62	0.14		
—6.0		1.16	1.13	1.06	0.94	0.77	0.53	0.27		
—8.0		0.38	0.37	0.35	0.32	0.27	0.21	0.14	0.06	
—8.8		0	0	0	0	0	0	0	0	0

The maximum stress in a shaft of triangular cross section, as evaluated from the model, occurs at the center of each side and is

$$S = 0.509 \frac{G\,\theta\,a}{L} \tag{31}$$

in which $\underline{a}$ is the altitude of the triangle.

This compares favorably with

$$S = \tfrac{1}{2} \frac{G\,\theta\,a}{L} \tag{31a}$$

from Timoshenko (4).

No theoretical results are available for comparison with the angle section, but the maximum stress will occur at the inside corner, the origin in Fig. 6. At that point, the stress is theoretically infinite if the corner is sharp, and decreases as the radius of fillet increases.

APPLICATION TO OTHER PROBLEMS

The use of the Laplaciometer may be extended to the solution of problems in which the Laplacian is not a constant. If the Laplacian is a variable and its value is known at a number of points throughout the area under consideration, the Laplaciometer may be used in the conventional manner except that the gage must be adjusted at each of the control points to give the proper reading.

The instrument may be used to solve problems in which the double Laplacian is known. For example, the general equation for the deflection of a plate, or slab, may be expressed as

$$\frac{\partial^4 w}{\partial x^4} + \frac{2\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{P(x,y)}{N} \tag{32}$$

in which $\underline{P(x,y)}$ is the load and $\underline{N}$ is the stiffness of the slab $\frac{Eh^3}{12(1-\mu^2)}$

The left hand side of the equation may be written as

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$

and hence is known as the double Laplacian.

Holl (3) has shown how this may be solved by establishing a new function $\underline{U}$, so that

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = U \tag{33}$$

Then Eq. (32) reduces to

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \frac{P(x,y)}{N} \tag{34}$$

Hence, the original problem may be solved by solving two successive Laplacian problems. Holl's procedure, which was illustrated by numerical methods may be employed in a solution using the Laplaciometer. Two models are required. The first, a $\underline{U}$ surface, may be obtained by the procedure outlined for the torsion problem letting $Q = \frac{P\,(x,y).}{N}$

Elevations of this surface are equal (or proportional to $\underline{U}$). The second surface, the $\underline{w}$ surface, is obtained as the solution to Eq. (33) by adjusting the height of the Laplaciometer to conform to values of $\underline{U}$ at the corresponding points.

The Laplaciometer may be used to solve equations of the type

$$\frac{\partial^2 z}{\partial x^2} = G\ (x) \tag{35}$$

by utilizing the center rod and two diagonally opposite rods. The other pair of diagonally opposite rods are clamped to remain fixed. The surface degenerates to a profile.

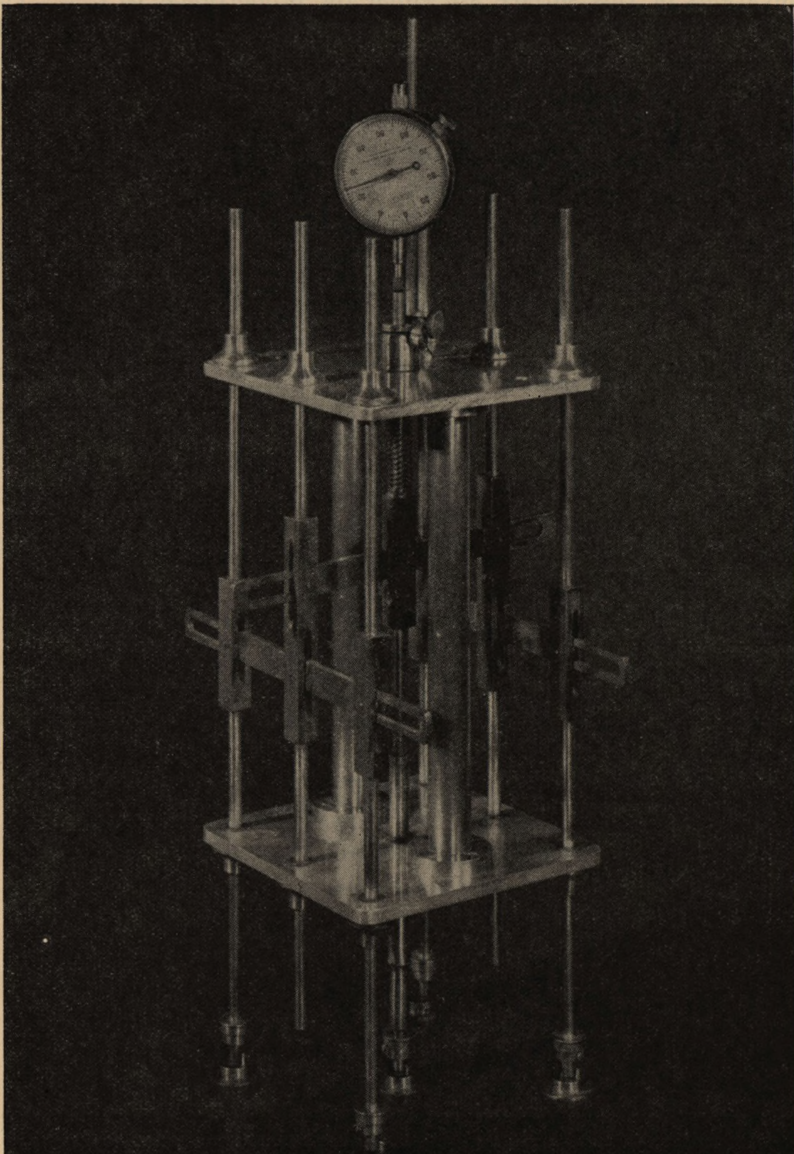
Equations of the form

$$\frac{\partial^2 z}{\partial x^2} + \frac{a\partial^2 z}{\partial y^2} = H\ (x,y) \tag{36}$$

may be solved by making the lengths of the arms A, B and C in Fig. 2 adjustable.

List of References

1. Griffith, A. and Taylor, C. Soap Films for Torsional Problems. Proc. Instn. Mech. Eng. 775-809, 1917
2. Nadai, A. Elastische Platten. J. Springer, Berlin
3. Holl, D. L., Analysis of Plate Problems by Difference Methods and the Superposition Principle, Journal of Applied Mechanics, Sept. 1936
4. Timoshenko, S. Theory of Elasticity, McGraw-Hill Book Co. 1934



The Laplaciometer. This instrument was constructed to fit a 3 in. square grid, and is 14 inches in height.

APPENDIX

Error of Approximation

If the elevation of the center point is

$$z_0 = f(x,y) \tag{1}$$

The elevations of the other points on the square grid of sides u are

$$\begin{aligned} z_1 &= f(x + u, y) \\ z_2 &= f(x, y + u) \\ z_3 &= f(x - u, y) \\ z_4 &= f(x, y - u) \end{aligned} \tag{2}$$

The expression for the Laplacian, Eq. (7), may be expanded by Taylor's theorem to give

$$\frac{z_1 + z_2 + z_3 + z_4 - 4z_0}{u^2} =$$

$$\frac{1}{u^2}\left[\begin{aligned} & f(x,y) + u\,f_x(x,y) + \frac{u^2}{2!}f_{xx}(x,y) + \frac{u^3}{2!}f_{xxx}(x,y) + \ldots\ldots \\ & + f(x,y) + u\,f_y(x,y) + \frac{u^2}{2!}f_{yy}'(x,y) + \frac{u^3}{2!}f_{yyy}(x,y) + \ldots\ldots \\ & + f(x,y) - u\,f_x(x,y) + \frac{u^2}{2!}f_{xx}(x,y) - \frac{u^3}{2!}f_{xxx}(x,y) + \ldots\ldots \\ & + f(x,y) - u\,f_y(x,y) + \frac{u^2}{2!}f_{yy}(x,y) - \frac{u^3}{2!}f_{yyy}(x,y) + \ldots\ldots \\ & - 4\,f(x,y) \end{aligned}\right] \tag{3}$$

in which the subscripts represent derivatives.

It is apparent that Eq. (3) reduces to

$$\nabla^2 z = \frac{z_1 + z_2 + z_3 + z_4 - 4z_0}{u^2}$$

$$= \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + \frac{2u^2}{4!}\left(\frac{\partial^4 z}{\partial x^4} + \frac{\partial^4 z}{\partial y^4}\right) + \frac{2u^4}{6!}\left(\frac{\partial^6 z}{\partial x^6} + \frac{\partial^6 z}{\partial y^6}\right) + \ldots \tag{4}$$

Hence, the error, the difference between Eq. (4) and the exact values of the Laplacian, is

$$E(x,y) = \frac{2u^2}{4!}\left(\frac{\partial^4 z}{\partial x^4} + \frac{\partial^4 z}{\partial y^4}\right) + \frac{2u^4}{6!}\left(\frac{\partial^6 z}{\partial x^6} + \frac{\partial^6 z}{\partial y^6}\right) + \ldots \tag{5}$$

The effect of the value of u, or size of the grid, upon the error is apparent.

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